1. What are the axioms of probability?

2. (a) What two definitions of probability supplement these axioms?

(b) What is Bayes’ Rule?

3. What is a valid argument?

4. What does it mean for two events (or propositions) to be mutually exclusive? To be independent?
5. Two events, A and B, are such that \( P(A) = 0.4 \), \( P(B) = 0.6 \), and \( P(A \& B) = 0.3 \)

(a) [2 points] Are A and B independent? Provide an argument or proof to support your answer.

(b) [2 points] Are A and B mutually exclusive? Provide an argument or proof to support your answer.

(c) [2 points] Are A and B exhaustive? Provide an argument or proof to support your answer.

Solve for the following:

(d) [2 points] \( P(A \lor B) \)

(e) [2 points] \( P(B|A) \)

(f) [2 points] \( P(\neg A) \)

(g) [2 points] \( P(\neg B) \)

(h) [2 points] \( P(\neg A \& \neg B) \)

*Hint:* Try using a Venn diagram to figure this out.
6. Being a connoisseur of donuts, Prof. Haber opens a donut shop, “Inductive Donuts.” Being a probability enthusiast, Prof. Haber decides that he will not reveal the donut menu, but instead require people to infer the donut varieties from random samplings of donuts. For every donut taken, Prof. Haber replaces it with the same variety.

Upon some rather delicious detective work, you’ve determined that 50% of the donuts in Prof. Haber’s shop are made with chocolate. You’ve also been able to determine that the probability that a donut will be frosted, given that it is chocolate, is 30%.

(a) What is the probability that a donut is both frosted and chocolate?

(b) Prof. Haber tells you that a donut being frosted or being made with chocolate are independent events. Given what you’ve already inferred, what is the probability that a donut in Prof. Haber’s shop is frosted?

(c) As you continue to sample Prof. Haber’s truly lovely donuts, you discover that whether a donut is glazed has no bearing on the probability of that donut having sprinkles (and vice-versa). Additionally, some donuts are neither glazed nor sprinkled. Assign a probabilities to a donut being glazed, and to a donut having sprinkles, that reflect these two facts. Prove that your probabilities reflect that these are independent but not exhaustive of the donuts.

(d) Every Inductive Donut is either baked or fried. No donut is both baked and fried, because Prof. Haber thinks that is an abomination. Assign probabilities that reflect these facts, and offer a proof of that.
7. Scandal hits Inductive Donuts! A diligent local reporter (perhaps a disgruntled former student?) discovers and reveals that some of the donuts are not, as claimed, made in shop, but out-sourced. They are fakes! 10% of jelly donuts, 20% of custard and (most shockingly) 80% of donut holes are not made in-house.

The reporter also reveals that of the donuts at Inductive Donuts, 20% are jelly, 10% custard, and 10% donut holes. These are exclusive of each other. No other donuts are involved in the scandal (i.e., all other donuts are made at the shop).

[6 points] (a) Your friend finds out the donut they just ate was not made in-house. Given that information, what is the probability your friend ate a donut hole? Demonstrate this using both Bayes’ Rule and the Probability Tree method.

[6 points] (b) Your friend can tell, with 70% accuracy, whether a custard donut is genuine or fake. Prof. Haber has said that if you can demonstrate that the probability that you’ve been given a fake donut is greater than 50%, that you will receive a refund.

You get a custard donut, which your friend says is a fake. You eat it anyways, then get another custard donut, which your friend again says is fake. What probability should you assign that you’ve received a fake donut? Is it enough to get a refund?
8. One of Ian Hacking's 'Odd Questions' concerns witnessing taxicabs. Here is the set up:

There are two kinds of taxis in town, Green and Blue. Green cabs dominate the market, with 85% of the cabs on the road. Like the Blue cabs, they are randomly distributed about town.

On a misty winter night, a taxi sideswiped another car and drove off. A witness says it was a blue cab. That witness is tested, and 80% of the time she correctly reports the color of the cab in conditions like those on the night of the accident. She was just as accurate in reporting a blue or green cab. Hacking demonstrated what probability we might assign to the cab being blue, given the report of the witness.

Here's the twist. You are on the jury, serving with other members of this class. The judge instructs you that may not conclude that the cab was one color or the other with confidence unless you can assign a probability of greater than 65% to that belief. You've heard from the first witness, and the prosecution says they have several more witnesses lined up can report, with the same level of accuracy, what color the cab was.

Starting from the data provided, how many witnesses will you need to see before you can assign a probability of greater than 65% to the cab being blue? Prove this using both Bayes' Rule and a Probability Tree.
[10 points] 9. A medical diagnostic test correctly reports the presence (or absence) of disease 99% of time. The overall rate of disease in the population is $\frac{1}{10,000}$.

What probability should a doctor assign that someone who tested positive has the disease? If the doctor runs the test again, and it comes out positive, what probability should she assign to that person having the disease?

Show your results using both Bayes’ Rule and the Probability Tree method.
10. Last night the St. Louis Cardinals (STL) defeated the Texas Rangers (TEX) in game 1 of Major League Baseball’s World Series. The first team to win four games wins the Series, and is crowned champion for the season. The teams are evenly matched, but playing at home gives each team a slight advantage. As a result, the probability of winning a home game is 60%. Game 2 will be played in St. Louis, games 3, 4, and 5 in Texas, and games 6 and 7 will be back in St. Louis.

(a) Assume that the St. Louis Cardinals win game 4. What is the probability of that win being the fourth win for STL in the Series? That is, given that STL wins game 4, what is the probability they swept the Series and are champions?

(Note: Remember that St. Louis has already won game 1!)

(b) If the Texas Rangers end up winning the world series, what is the probability that they won their fourth game of the Series in game 6 of the World Series? That is, given that the Texas Rangers won the world series, what is the probability that they did so by winning four of the next five games?
[10 points] 11. Abigail, Beauregard, Chisolm, and Darlene are finalists on *The Worlds Worst Reality TV Show*. *The Worlds Worst Reality TV Show* has three remaining possible challenges. Which challenge comes up is determined by the drawing of straws (see below). Abigail won the previous challenge, so has an advantage going into the final rounds: she gets to draw straws first to determine which challenge she will play in the first round, and can then select her opponent for that round. The other two remaining challengers will face off, and the winners of each challenge will then play in the championship round.

In other words, if Abigail wins the first round, she will then face one of the other two competitors not faced before. The championship round challenge will be an invigorating game of ‘wff ’n proof’.

Below is a table with the following information. It tells you the probability of any one challenge being selected for the first round, and the probability of Abigail beating her competitors in each challenge. It also provides you with information about the final challenge.

<table>
<thead>
<tr>
<th>American Idol Outtakes</th>
<th>I ate WHAT?!</th>
<th>Win Ayn Rand’s Money!</th>
<th>wff ’n proof (final round)</th>
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<tbody>
<tr>
<td>Probability of Being Selected for first round</td>
<td>30%</td>
<td>50%</td>
<td>20%</td>
</tr>
<tr>
<td>Beauregard</td>
<td>60%</td>
<td>70%</td>
<td>40%</td>
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<tr>
<td>Chisolm</td>
<td>40%</td>
<td>60%</td>
<td>50%</td>
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<tr>
<td>Darlene</td>
<td>70%</td>
<td>50%</td>
<td>30%</td>
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Finally, the probability of Beauregard beating Chisolm in any challenge is 60%, and of beating Darlene is 30%. Darlene has a 50% probability of beating Chisolm in any challenge.

(a) What is the probability that Abigail will win *The Worlds Worst Reality TV Show*?

(b) If Abigail wins *The Worlds Worst Reality TV Show*, what is the probability that she faced Beauregard in the final rounds?

(c) Assume Abigail faced Darlene in the first round. What is the probability that Darlene faced Chisolm in the final round if Darlene ultimately won *The Worlds Worst Reality TV Show*?

(*Hint:* If you use %’s in your calculations, rounding up to whole numbers is allowed.)
12. [10 (bonus)] Given the axioms of probability, the definition of independence, conditional probability, and the rules of logic and algebra, demonstrate how the following version of Bayes’ Rule may be derived:

\[ P(X|Y) = \frac{P(X) \cdot P(Y|X)}{P(X) \cdot P(Y|X) + P(\sim X) \cdot P(Y|\sim X)} \]
<table>
<thead>
<tr>
<th>Question</th>
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<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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Page 10 of 10