1. (a) What are all the possible \textit{atomic propositions} or \textit{states} of Boolean algebra $B_1$ consisting of two atomic sentence letters ($X, Y$)?

\textbf{Solution:} 
\begin{align*}
  s_1 &: X \land Y \\
  s_2 &: X \land \neg Y \\
  s_3 &: \neg X \land Y \\
  s_4 &: \neg X \land \neg Y
\end{align*}

(b) Display the possible logical space of $B_1$ using a truth table.

(c) Display the \textit{probability} space $M$ of $B_1$ using a truth table and a Venn diagram.

For the rest of the exam this will be $M_1$.

\begin{tabular}{ccc|c}
\hline
$X$ & $Y$ & $s_i$ & $Pr(s_i)$ \\
\hline
T  & T  & $s_1$ & $a_1$ \\
T  & F  & $s_2$ & $a_2$ \\
F  & T  & $s_3$ & $a_3$ \\
F  & F  & $s_4$ & $a_4$ \\
\hline
\end{tabular}

I don’t know how to draw a Venn diagram in LaTeX yet, so you’ll have to figure it out on your own or ask me.

2. For Boolean algebra $B_2$ consisting of three atomic sentence letters ($X, Y, Z$):

(a) Display the possible logical space of $B_2$ using a truth table.

(b) Display the \textit{probability} space $M$ of $B_2$ using a truth table and a Venn diagram.

For the rest of the exam this will be $M_2$.

\begin{tabular}{ccc|c}
\hline
$X$ & $Y$ & $Z$ & $s_i$ & $Pr(s_i)$ \\
\hline
T  & T  & T  & $s_1$ & $a_1$ \\
T  & T  & F  & $s_2$ & $a_2$ \\
T  & F  & T  & $s_3$ & $a_3$ \\
T  & F  & F  & $s_4$ & $a_4$ \\
F  & T  & T  & $s_5$ & $a_5$ \\
F  & T  & F  & $s_6$ & $a_6$ \\
F  & F  & T  & $s_7$ & $a_7$ \\
F  & F  & F  & $s_8$ & $a_8$ \\
\hline
\end{tabular}

I don’t know how to draw a Venn diagram in LaTeX yet, so you’ll have to figure it out on your own or ask me.
3. Use a truth table to assign $Pr_M(s_i)$ to all the states in $B_2$ such that:
$Pr(X) = 0.6$, $Pr(Y) = 0.3$, $Pr(Z) = 0.3$, $Pr(X \& Y \& Z) = 0.1$, and $Pr(Y \& Z) = 0.2$.
Designate $Pr_M(s_i)$ for all unspecified states such that the assignments are coherent.
For the rest of this exam this will be $M_3$.

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
<th>Z</th>
<th>s_i</th>
<th>$Pr(s_i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>$s_1$</td>
<td>0.1</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>$s_2$</td>
<td>0.1</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>$s_3$</td>
<td>0.1</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>$s_4$</td>
<td>0.3</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>$s_5$</td>
<td>0.1</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>$s_6$</td>
<td>0</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>$s_7$</td>
<td>0</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>$s_8$</td>
<td>0.3</td>
</tr>
</tbody>
</table>

**Solution:**

Other solutions are also possible. You just have to make sure that your total probability sums to 1.0, and that the values for the designated probabilities add up correctly. E.g., $Pr(X) = \sum_{i=1-4} Pr(s_i)$, which in this case should add to 0.6.

4. What is the probability of a proposition $p$ in a probability model $M$?

**Solution:**

The probability of a proposition $p$ is equal to the sum of probabilities of states $s_i$ that entail the truth of proposition $p$;

$Pr(p) = \sum_{s_i \Rightarrow p} Pr(s_i)$

5. Determine the probability of the following propositions in $M_1$

(a) $\sim X \& Y$

**Solution:** $Pr(\sim X \& Y) = Pr(s_3) = a_3$

(b) $X \vee Y$

**Solution:** $Pr(X \vee Y) = Pr(s_1) + Pr(s_2) + Pr(s_3) = a_1 + a_2 + a_3$
6. Determine the probability of the following propositions in $M_2$ & $M_3$

[2 points] (a) $\sim(X \lor Y)$

**Solution:**

$M_2$: $Pr(\sim(X \lor Y)) = Pr(s_7) + Pr(s_8) = a_7 + a_8$

$M_3$: $Pr(\sim(X \lor Y)) = Pr(s_7) + Pr(s_8) = 0 + 0.3 = 0.3$

*Note:* your values for $M_3$ may differ depending on your answer to question 3

[2 points] (b) $(X \lor Y) \land Z$

**Solution:**

$M_2$: $Pr((X \lor Y) \land Z) = Pr(s_1) + Pr(s_3) + Pr(s_5) = a_1 + a_3 + a_5$

$M_3$: $Pr((X \lor Y) \land Z) = Pr(s_1) + Pr(s_3) + Pr(s_5) = 0.1 + 0.1 + 0.1 = 0.3$

*Note:* your values for $M_3$ may differ depending on your answer to question 3

[5 points] 7. Prove, algebraically, that in $M_2$: $Pr(\sim[X \land (\sim Y \lor \sim Z)]) = Pr(\sim X) + Pr(Y \land Z) - Pr(\sim X \land Y \land Z)$

There are a number of strategies available here, but I'd recommend first solving for each $p$ on a truth table:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$Y$</th>
<th>$Z$</th>
<th>$\sim[X \land (\sim Y \lor \sim Z)]$</th>
<th>$\sim X$</th>
<th>$Y \land Z$</th>
<th>$\sim X \land Y \land Z$</th>
<th>$s_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>$s_1$</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>$s_2$</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>$s_3$</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>$s_4$</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>$s_5$</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>$s_6$</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>$s_7$</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>$s_8$</td>
</tr>
</tbody>
</table>

**Solution:**

Now simply solve for $Pr(p)$ for each $p$ (First step coded by color):

$Pr(\sim[X \land (\sim Y \lor \sim Z)]) = Pr(\sim X) + Pr(Y \land Z) - Pr(\sim X \land Y \land Z)$

$a_1 + a_5 + a_6 + a_7 + a_8 = a_5 + a_6 + a_7 + a_8 + a_1 + a_5 - a_5^*$

$a_1 + a_5 + a_6 + a_7 + a_8 = a_5 + a_6 + a_7 + a_8 + a_1 + a_5 - a_5^*$

$a_1 + a_5 + a_6 + a_7 + a_8 = a_1 + a_5 + a_6 + a_7 + a_8$

* I’ve skipped the ‘$Pr(s_i) \iff a_i$’ step, which I assume you understand at this point.
[5 points] 8. Using tools from the algebraic proof methods, prove that if \( \sim X \lor Y \) is true, then \( \Pr(X) \leq \Pr(Y) \) must also be true.

(Hint: If you are given the truth value of a proposition, consider what you may infer about \( Pr_M(s_i) \) for all possible \( s_i \) in \( M \).)

**Solution:** First let’s take a look at a truth table describing the logical & probability space if \( \sim X \lor Y \) is true:

<table>
<thead>
<tr>
<th>( X )</th>
<th>( Y )</th>
<th>( \sim X \lor Y )</th>
<th>( s_i )</th>
<th>( Pr(s_i) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>( s_1 )</td>
<td>( a_1 )</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>( s_2 )</td>
<td>0</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>( s_3 )</td>
<td>( a_3 )</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>( s_4 )</td>
<td>( a_4 )</td>
</tr>
</tbody>
</table>

Because \( \sim X \lor Y \) is false in \( s_2 \), we know that if \( \sim X \lor Y \) is true then \( Pr(s_2) = 0 \). We don’t know anything about the \( Pr(s_i) \) for the other states, except that together they must sum to 1.0.

Now, using the algebraic method, solve for \( Pr(X) \) & \( Pr(Y) \).

\[
Pr(X) = Pr(s_1) + Pr(s_2) = a_1 + 0 = a_1;
\]

\[
Pr(Y) = Pr(s_1) + Pr(s_3) = a_1 + a_3;
\]

\( a_1 \leq a_1 + a_3 \), so \( Pr(X) \leq Pr(Y) \)

QED
A probability model $M$ is a Boolean algebra of propositions $B$ together with a probability function $Pr_M(s_i)$ satisfying the following three axioms:

1. **Normativity**
   For any proposition $p$, the probability of that proposition is non-negative;
   \[ \forall p, Pr(p) \geq 0 \]

2. **Certainty**
   All tautologies $\top$ have a probability of 1;
   \[ Pr(\top) = 1; \]
   Where $\Omega$ is a necessary event, $Pr(\Omega) = 1$

3. **Additivity**
   The probability of a disjunction of mutually exclusive (m.e.) propositions is the sum of the probability of its disjunctions;
   \[ Pr(p \lor q) = Pr(p) + Pr(q) \]

3 alt. **Additivity** (General Form)
   The probability of a disjunction of propositions is the sum of the probability of its disjuncts minus the conjunction of those disjuncts;
   \[ Pr(p \lor q) = Pr(p) + Pr(q) - Pr(p \land q) \]

Additionally, we import the rules of logic and algebra and get the following two definitions:

**Conditional Probability:** $Pr(X|Y) = \frac{Pr(X \land Y)}{Pr(Y)}$

**Independence:** $X$ and $Y$ are independent of each other iff $Pr(X \land Y) = Pr(X) \cdot Pr(Y)$

9. Prove, axiomatically, that $Pr(\sim p) = 1 - Pr(p)$. This will be theorem 4.

**Solution:**

**Proof:**

- By logic, $p \lor \sim p$ is a tautology $\top$;
- By (2), $Pr(p \lor \sim p) = 1$;
- By logic, $p$ and $\sim p$ are mutually exclusive;
- By (3) $Pr(p \lor \sim p) = Pr(p) + Pr(\sim p)$;
- By algebra, $Pr(p) + Pr(\sim p) = 1$;
- By algebra, $Pr(\sim p) = 1 - Pr(p)$
  \[ \text{QED} \]
For 10 you may appeal to any of the following theorems by number:

(4) \( Pr(\sim X) = 1 - Pr(X) \);
(5) \( Pr(\bot) = 0 \) (where \( \bot \) is a contradiction or logical falsehood);
(6) If \( X \equiv Y \) or \( X \iff Y \), then \( Pr(X) = Pr(Y) \);
(7) \( Pr(X) = Pr(X\&Y) + Pr(X\&\sim Y) \)

[10 points] 10. Prove, axiomatically, that if \( p \) entails \( q \), then \( Pr(p) \) is less than or equal to \( Pr(q) \);
That is, prove:
If \( p \Rightarrow q \), then \( Pr(p) \leq Pr(q) \)

(Hint: Theorem (7) tells us that:
\( Pr(p) = Pr(p\&q) + Pr(p\&\sim q) \)
\( Pr(q) = Pr(p\&q) + Pr(\sim p\&q) \)
Keep in mind that these are general forms, much like the general form of the Axiom of Additivity. Consider whether a special case of either of these may be in order, akin to the special case of the Axiom of Additivity.)

Solution: There are several solutions. Let’s look at two.

Proof 1: If \( p \Rightarrow q \) . . .
- By definition, if \( p \Rightarrow q \), then if \( p \) is true, \( q \) must also be true, i.e., if \( p \) is true, then \( p\&q \) must be true as well;
- By logic, \( p\&q \) is mutually exclusive of \( p\&\sim q \). So if \( p \Rightarrow q \) is true and \( p \) is true, then \( p\&\sim q \)
must be false, i.e., if \( p \) is true, we may treat \( p\&\sim q \) as a \( \bot \);
- By (7) \( Pr(p) = Pr(p\&q) + Pr(p\&\sim q) \), but by (5) \( Pr(p\&\sim q) = 0 \) so
- \( Pr(p) = Pr(p\&q) + 0 = Pr(p\&q) \);
- By (7) \( Pr(q) = Pr(p\&q) + Pr(\sim p\&q) \);
- By algebra we can substitute \( Pr(p\&q) \) for \( Pr(p) \), giving us \( Pr(q) = Pr(p) + Pr(\sim p\&q) \);
- By (1) \( Pr(\sim p\&q) \geq 0 \), so, by algebra, \( Pr(q) \geq Pr(p) \);
QED

Proof 2:
- Prove that if \( p \Rightarrow q \), then \( Pr(p) \leq Pr(q) \)
- By (7) \( Pr(p) = Pr(p\&q) + Pr(p\&\sim q) \), and \( Pr(q) = Pr(p\&q) + Pr(\sim p\&q) \),
so we need to prove that if \( p \Rightarrow q \), \( Pr(p\&q) + Pr(p\&\sim q) \leq Pr(p\&q) + Pr(\sim p\&q) \);
- By algebra we can subtract \( Pr(p\&q) \) from each side, leaving us with the need to prove
that, if \( p \Rightarrow q \), then \( Pr(p\&\sim q) \leq Pr(\sim p\&q) \);
- By definition, if \( p \Rightarrow q \) is true, then whenever \( p \) is true, \( q \) must be true as well. That is,
whenever \( p \) is true, \( p\&q \) will be true as well.
- By logic, \( p\&q \) and \( p\&\sim q \) are mutually exclusive, so if \( p \Rightarrow q \) is true, then \( p\&\sim q \) will
always be false, and may be treated as a \( \bot \);
- By (5) \( Pr(p\&\sim q) = 0 \);
- By (1) \( Pr(\sim p\&q) \geq 0 \);
- By algebra, then, \( Pr(p\&\sim q) \leq Pr(\sim p\&q) \), so \( Pr(p) \leq Pr(q) \)
QED
Extra Credit: Prove, axiomatically, that if \( p \Rightarrow q \), then \( Pr(q|p) = 1 \).

**Solution:** Proof:

- If \( p \Rightarrow q \), then if \( p \) is true \( q \) must be true as well;
- By logic, If \( p \Rightarrow q \), then \( p \iff q \& p \);
- By (6) \( Pr(p) = Pr(q \& p) \);
- By definition, \( Pr(q|p) = \frac{Pr(q \& p)}{Pr(p)} \);
- By algebra, \( Pr(q|p) = \frac{Pr(p)}{Pr(p)} = 1 \);

QED