Supplemental Homework - Answers  
(Due in class, Thursday Dec. 8)  

Section 1 – Probability Space  

1. What are all the possible atomic propositions or states of: 
   (a) Boolean algebra 1 ($B_1$) consisting of two atomic sentence-letters ($X, Y$)?  
   
   $s_1: X \& Y$  
   $s_2: X \& \neg Y$  
   $s_3: \neg X \& Y$  
   $s_4: \neg X \& \neg Y$  

   (b) $B_2$ consisting of three atomic sentence-letters ($X, Y, Z$)?  
   
   $s_5: X \& Y \& Z$  
   $s_6: X \& Y \& \neg Z$  
   $s_7: X \& \neg Y \& Z$  
   $s_8: X \& \neg Y \& \neg Z$  

2. Display the possible logical space of $B_1$ & $B_2$ using truth tables and Venn diagrams.  

   **$B_1$:**  
   
   \[
   \begin{array}{c|c|c|c}
   X & Y & \text{States ($s_i$)} \\
   \hline
   T & T & s_1 \\
   T & F & s_2 \\
   F & T & s_3 \\
   F & F & s_4 \\
   \end{array}
   \]

   **$B_2$:**  
   
   \[
   \begin{array}{c|c|c|c|c}
   X & Y & Z & \text{States ($s_i$)} \\
   \hline
   T & T & T & s_1 \\
   T & T & F & s_2 \\
   T & F & T & s_3 \\
   T & F & F & s_4 \\
   F & T & T & s_5 \\
   F & T & F & s_6 \\
   F & F & T & s_7 \\
   F & F & F & s_8 \\
   \end{array}
   \]
3. What is a probability model \((M)\)?

A probability model \((M)\) is an extension of a Boolean algebra \((B)\), where the states of \(B\) are assigned probability values determined by some probability function \(\Pr(s)\), where \(\sum_\text{Pr}(s) = 1\).

4. We learned that Probability Models \((M)\) may be conceptualized as extensions of Boolean Algebras \((B)\). Consider a \(B\) with two atomic sentence letters \((X, Y)\), and four states:

\(s_1\): \((X \& Y)\)
\(s_2\): \((X \& \sim Y)\)
\(s_3\): \((\sim X \& Y)\)
\(s_4\): \((\sim X \& \sim Y)\)

(a) Display the probability space of \(M\) of \(B\) using truth tables and Venn diagrams. For the rest of the homework assignment, this will be \(M_1\).

\(M_1:\)

\[\begin{array}{ccc|c}
 X & Y & \text{States (s)} & \Pr(s) \\
 \hline
 T & T & s_1 & a_1 \\
 T & F & s_2 & a_2 \\
 F & T & s_3 & a_3 \\
 F & F & s_4 & a_4 \\
\end{array}\]

(b) Display the probability space of \(M\) of \(B\) using truth tables and Venn diagrams when the values of \(\Pr_M(s)\) are:

\(\Pr_M(s_1) = 1/6\)
\(\Pr_M(s_2) = 1/4\)
\(\Pr_M(s_3) = 1/8\)
\(\Pr_M(s_4) = 11/24\)

For the rest of the homework assignment, this will be \(M_3\).

\(M_3:\)

\[\begin{array}{ccc|c}
 X & Y & \text{States (s)} & \Pr(s) \\
 \hline
 T & T & s_1 & 1/6 \\
 T & F & s_2 & 1/4 \\
 F & T & s_3 & 1/8 \\
 F & F & s_4 & 11/24 \\
\end{array}\]
(e) Display the probability space of \(M\) of \(B\) using truth tables and Venn diagrams when the values of \(\text{Pr}_M(s)\) are:
\[
\begin{align*}
\text{Pr}_M(s_1) &= 0 \\
\text{Pr}_M(s_2) &= 1/3 \\
\text{Pr}_M(s_3) &= 1/6 \\
\text{Pr}_M(s_4) &= 1/2
\end{align*}
\]
For the rest of the homework assignment, this will be \(M_4\).

\[
M_4:
\begin{array}{c|c|c|c}
X & Y & \text{States (s)} & \text{Pr}(s) \\
\hline
T & T & s_1 & 0 \\
T & F & s_2 & 1/3 \\
F & T & s_3 & 1/6 \\
F & F & s_4 & 1/2 \\
\end{array}
\]

(d) Display the probability space of \(M\) of \(B_2\) from problem 1b above using truth tables and Venn diagrams. For the rest of the homework assignment, this will be \(M_2\).

\[
M_2:
\begin{array}{c|c|c|c|c}
X & Y & Z & \text{States (s)} & \text{Pr}(s) \\
\hline
T & T & T & s_1 & a_1 \\
T & T & F & s_2 & a_2 \\
T & F & T & s_3 & a_3 \\
T & F & F & s_4 & a_4 \\
F & T & T & s_5 & a_5 \\
F & T & F & s_6 & a_6 \\
F & F & T & s_7 & a_7 \\
F & F & F & s_8 & s_8 \\
\end{array}
\]

5. (a) What is the relation between \(X\) and \(Y\) in \(M_4\)?

\(X\) and \(Y\) are exclusive; both cannot be true.

(b) Assign probabilities to the states of \(M_1\) such that \(X\) and \(Y\) will be exclusive and exhaustive. For the rest of the homework assignment this will be \(M_5\).

(see below)

(c) Draw a truth table and Venn diagram displaying the probability space of \(M_5\) in 5b.
The table $M^*_Y$ is shown below:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$Y$</th>
<th>States ($s_i$)</th>
<th>$\Pr(s_i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>$s_1$</td>
<td>0</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>$s_2$</td>
<td>1/3</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>$s_3$</td>
<td>2/3</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>$s_4$</td>
<td>0</td>
</tr>
</tbody>
</table>

The relationship between $X$, $Y$, and $s_i$ is also depicted in the diagram on the right.
Section 2 – Algebraic Proofs

6. (a) What is the probability of a proposition \( p \) in a probability model \( M \)?

The probability of a proposition \( p \) is equal to the sum of probabilities of states \( (s_i) \) that entail the truth of proposition \( p \).

\[
\Pr(p) = \sum_{i:p} \Pr(s_i)
\]

(b) What is the probability of \( (X \lor Y) \) in \( M_1 \)?

\( (X \lor Y) \) is true in states \( s_1, s_2, \) and \( s_3 \). So \( \Pr(X \lor Y) = \Pr(s_1) + \Pr(s_2) + \Pr(s_3) = a_1 + a_2 + a_3 \).

(c) What is the probability of \( (X \lor Y) \) in \( M_3 \)?

\( (X \lor Y) \) is true in states \( s_1, s_2, \) and \( s_3 \). So \( \Pr(X \lor Y) = \Pr(s_1) + \Pr(s_2) + \Pr(s_3) = a_1 + a_2 + a_3 \).

In \( M_3 \), this would be \( 1/6 + 1/4 + 1/8 = 4/24 + 6/24 + 3/24 = 13/24 \).

(d) What is the probability of \( (X \lor Y) \) in \( M_4 \)?

\( (X \lor Y) \) is true in states \( s_1, s_2, \) and \( s_3 \). So \( \Pr(X \lor Y) = \Pr(s_1) + \Pr(s_2) + \Pr(s_3) = a_1 + a_2 + a_3 \).

In \( M_4 \), this would be \( 0 + 1/3 + 1/6 = 2/6 + 1/6 = 3/6 = 1/2 \).

(e) What is the probability of \( (X \lor Y) \) in \( M_2 \)?

To answer this, look back at answer 2 – the logical space for \( B_2 \). In which states is \( (X \lor Y) \) true? States \( s_1, s_2, s_3, s_4, s_5, \) and \( s_6 \). So \( \Pr(X \lor Y) = \Pr(s_1) + \Pr(s_2) + \Pr(s_3) + \Pr(s_4) + \Pr(s_5) + \Pr(s_6) = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 \).

7. Determine the probability of the following propositions \( p \) in \( M_1, M_3, \) and \( M_4 \):

(a) \( (X \land Y) \)

\[
\begin{align*}
M_1: \Pr(X \land Y) &= \Pr(s_1) = a_1 \\
M_3: \Pr(X \land Y) &= \Pr(s_1) = a_1 = 1/6 \\
M_4: \Pr(X \land Y) &= \Pr(s_1) = a_1 = 0
\end{align*}
\]

(b) \( (\neg X \lor Y) \)

\[
\begin{align*}
M_1: \Pr(\neg X \lor Y) &= \Pr(s_1) + \Pr(s_3) + \Pr(s_5) = a_1 + a_3 + a_4. \\
M_3: \Pr(\neg X \lor Y) &= \Pr(s_1) + \Pr(s_3) + \Pr(s_5) = a_1 + a_3 + a_4 = 1/6 + 1/8 + 11/24 = 18/24 = 3/4 \\
M_4: \Pr(\neg X \lor Y) &= \Pr(s_1) + \Pr(s_3) + \Pr(s_5) = a_1 + a_3 + a_4 = 0 + 1/6 + 1/2 = 4/6 = 2/3
\end{align*}
\]
(c) \((X \rightarrow Y) \& Y\)

(you may need to make a truth table to determine which states entail the truth of \((X \rightarrow Y) \& Y\).

\[ M_1: \Pr((X \rightarrow Y) \& Y) = \Pr(s_1) + \Pr(s_3) = a_1 + a_3. \]
\[ M_2: \Pr((X \rightarrow Y) \& Y) = \Pr(s_1) + \Pr(s_3) = a_1 + a_3 = 1/6 + 1/8 = 7/24 \]
\[ M_3: \Pr((X \rightarrow Y) \& Y) = \Pr(s_1) + \Pr(s_3) = a_1 + a_3 = 0 + 1/6 = 1/6 \]

8. Determine the probability of the following propositions \((p)\) in \(M_2:\)

(a) \((X \lor Y)\)

\[ \Pr(X \lor Y) = \Pr(s_1) + \Pr(s_2) + \Pr(s_3) + \Pr(s_4) + \Pr(s_5) + \Pr(s_6) = a_1 + a_2 + a_3 + a_4 + a_5 + a_6. \]

(b) \((X \& Y \& Z)\)

\[ \Pr(X \& Y \& Z) = \Pr(s_1) = a_1. \]

(c) \((X \lor Y) \rightarrow Z\)

\[ \Pr((X \lor Y) \rightarrow Z) = \Pr(s_1) + \Pr(s_3) + \Pr(s_5) + \Pr(s_6) = a_1 + a_3 + a_5 + a_6. \]

(d) \((X \lor Y) \& (Y \lor Z)\)

\[ \Pr((X \lor Y) \& (Y \lor Z)) = \Pr(s_1) + \Pr(s_2) + \Pr(s_3) + \Pr(s_6) = a_1 + a_2 + a_3 + a_6. \]

9. Prove, algebraically, that \(\Pr(X \lor Y) = \Pr(X) + \Pr(Y) - \Pr(X \& Y)\) in …

(a) \(M_1:\)

\[ \Pr(X \lor Y) = \Pr(X) + \Pr(Y) - \Pr(X \& Y) \]
\[ \Pr(s_1) + \Pr(s_2) + \Pr(s_3) = [\Pr(s_1) + \Pr(s_2)] + [\Pr(s_1) + \Pr(s_3)] - \Pr(s_1) \]
\[ a_1 + a_2 + a_3 = a_1 + a_2 + a_1 + a_3 - a_1 \]
\[ a_1 + a_2 + a_3 = a_1 + a_2 + a_3 \]

(b) \(M_3:\)

Proof is same as above. Note, though, that this is a special case of (a). \(\Pr(X \& Y) = 0\), and may be dropped. This is an example of general cases where \(X\) and \(Y\) are mutually exclusive. When this condition is met:

\[ \Pr(X \lor Y) = \Pr(X) + \Pr(Y) \]
10. Prove, algebraically, that: \( \Pr(X) = \Pr(X \& Y) + \Pr(X \& \sim Y) \) in \( M_1 \).

\[
\Pr(X) = \Pr(X \& Y) + \Pr(X \& \sim Y) \\
\Pr(s_1) + \Pr(s_2) = \Pr(s_1) + \Pr(s_2)
\]

11. Define \( \Pr(X|Y) \) in a non-conditional form.

\[
\Pr(X|Y) = \frac{\Pr(X \& Y)}{\Pr(Y)} = \Pr\left(\frac{X \& Y}{Y}\right)
\]

12. Prove, algebraically, that: \( \Pr(X|Y) = \frac{\Pr(Y|X) \cdot \Pr(X)}{\Pr(Y)} \) in \( M_1 \).

\[
\Pr(X|Y) = \frac{\Pr(Y|X) \cdot \Pr(X)}{\Pr(Y)} \\
\frac{\Pr(X \& Y)}{\Pr(Y)} = \frac{\Pr(Y \& X) \cdot \Pr(X)}{\Pr(Y)}
\]

13. Prove, algebraically, that: \( \Pr(X \lor Y|Z) = \Pr(X|Z) + \Pr(Y|Z) - \Pr(X \& Y|Z) \) in \( M_2 \).

\[
\Pr((X \lor Y) \& Z) = \Pr(X \& Z) + \Pr(Y \& Z) - \Pr(X \& Y \& Z) \\
\frac{\Pr(s_1) + \Pr(s_2) + \Pr(s_3)}{\Pr(s_1) + \Pr(s_2) + \Pr(s_3)} = \frac{[\Pr(s_1) + \Pr(s_2)] + [\Pr(s_1) + \Pr(s_3) - \Pr(s_2)]}{\Pr(s_1) + \Pr(s_2) + \Pr(s_3)}
\]

\[
\frac{a_1 + a_4 + a_5}{a_1 + a_4 + a_5 + a_7} = \frac{a_1 + a_4 + a_5 - a_7}{a_1 + a_4 + a_5 + a_7}
\]

\[
\frac{a_1 + a_4 + a_5}{a_1 + a_4 + a_5 + a_7} = \frac{a_1 + a_4 + a_5}{a_1 + a_4 + a_5 + a_7}
\]

14. What is Bayes Rule?

\[
\Pr(X|Y) = \frac{\Pr(Y|X) \cdot \Pr(X)}{\Pr(Y)}
\]
15. Prove, algebraically, that \( \Pr(x \& y) = \Pr(x) \cdot \Pr(y) \iff \Pr(x|y) = \Pr(x|\neg y) \).

(hint: Do this in \( M_1 \); start by determining what each sentence is algebraically, then what each side of the biconditional would look like. Your proof should show that the left and right side of the biconditional are logically equivalent.)

First, let’s algebraically define some of the components:

\[
\Pr(x \& y) = \Pr(s_i) = a_i \\
\Pr(x) \cdot \Pr(y) = [\Pr(s_i) + \Pr(s_j)] \cdot [\Pr(s_i) + \Pr(s_3)] \\
= (a_1 + a_2) \cdot (a_1 + a_3) \\
\Pr(x|y) = \Pr(x \& y)/\Pr(y) \\
= \Pr(s_i)/[\Pr(s_i) + \Pr(s_j)] \\
= a_i/(a_1 + a_3) \\
\Pr(x|\neg y) = \Pr(x \& \neg y)/\Pr(\neg y) \\
= \Pr(s_j)/[\Pr(s_2) + \Pr(s_4)] \\
= a_2/(a_2 + a_4) \\
= a_2/[1 - (a_1 + a_3)]
\]

Now let’s start doing some algebra:

\[
a_1 = (a_1 + a_2) \cdot (a_1 + a_3) \iff \frac{a_1}{(a_1 + a_3)} = \frac{a_2}{(a_2 + a_4)} \\
\iff a_1 \cdot (a_2 + a_4) = a_2 \cdot (a_1 + a_3) \\
\iff a_1 \cdot (1 - (a_1 + a_3)) = a_2 \cdot (a_1 + a_3) \\
\iff a_1 - a_1 \cdot (a_1 + a_3) = a_2 \cdot (a_1 + a_3) \\
\iff a_1 = a_1 \cdot (a_1 + a_3) + a_2 \cdot (a_1 + a_3) \\
\iff a_1 = (a_1 + a_2) \cdot (a_1 + a_3) \iff a_1 = (a_1 + a_2)\cdot(a_1 + a_3)
\]

Section 3 – Axiomatic Proofs

A probability model \( (M) \) is a Boolean algebra\(^1\) of propositions \( B \) together with a probability function \( \Pr(\cdot) \) satisfying the following three axioms:

1. For any proposition, the probability of that proposition is non-negative; 
   \( \forall p, \Pr(p) \geq 0 \)

2. All tautologies \( (T) \) have a probability of 1; 
   \( \Pr(T) = 1 \); 
   where \( \Omega \) is a logical truth, \( \Pr(\Omega) = 1 \)

3. The probability of a disjunction of propositions is the sum of the probability of its disjuncts minus the probability of the conjunction of its disjuncts: 
   \( \Pr(p \lor q) = \Pr(p) + \Pr(q) - \Pr(p \land q) \)

\(^1\) This means we get to import the rules of logic and algebra.
A special case of this axiom concerns the disjunction of mutually exclusive (m.e.) propositions, in which case the probability is simply the sum of the probability of its disjuncts (why?)

\[
\Pr(p \lor q) = \Pr(p) + \Pr(q)
\]

**Def'n:**

\[
\Pr(X \mid Y) = \frac{\Pr(X \land Y)}{\Pr(Y)} = \Pr\left(\frac{X \land Y}{Y}\right)
\]

16. Prove, axiomatically, that \(\Pr(\neg p) = 1 - \Pr(p)\). This will be theorem 4.

**Proof:** Since \(p \lor \neg p\) is a tautology, then by (2) \(\Pr(p \lor \neg p) = 1\); and since \(p\) and \(\neg p\) are mutually exclusive, then by (3) \(\Pr(p \lor \neg p) = \Pr(p) + \Pr(\neg p)\). Together, this implies that \(\Pr(p) + \Pr(\neg p) = 1\), and, by algebra, \(\Pr(\neg p) = 1 - \Pr(p)\).

Q.E.D.

17. Prove, axiomatically, that the probability of a logical falsity is 0, i.e., \(\Pr(F) = 0\).

**Proof:** By logic, a logical falsity \((F)\) is equivalent to the negation of a tautology \((T)\). Since all states that entail \(\neg T\) will also entail \(F\), this implies that \(\Pr(\neg T) = \Pr(F)\). By (4), \(\Pr(\neg T) = 1 - \Pr(T)\), so \(\Pr(F) = 1 - \Pr(T)\). By (2) \(\Pr(T) = 1\), so \(\Pr(F) = 1 - 1\), which implies that \(\Pr(F) = 0\).

Q.E.D.

18. Prove, axiomatically, that if \(x\) and \(y\) mutually entail each other, then \(\Pr(x) = \Pr(y)\).

**Proof:** Assume that \(x\) and \(y\) mutually entail each other. Then, by logic, \(x\) and \(\neg y\) are mutually exclusive and exhaustive which implies that \(x \lor \neg y \iff T\). By (2) \(\Pr(x \lor \neg y) = 1\), and by (3) \(\Pr(x) + \Pr(\neg y) = 1\). By (4) \(\Pr(x) + (1 - \Pr(y)) = 1\). By algebra (subtract \((1 - \Pr(y)) from\ each\ side), \(\Pr(x) = \Pr(y)\), when \(x\) and \(y\) mutually entail one another (i.e, are logically equivalent).

Q.E.D.

19. Prove, axiomatically, that \(\Pr(x) = \Pr(x \land y) + \Pr(x \land \neg y)\)

(hint: \(x \land T\) is logically equivalent to \(x\). Why?)

**Proof:** Assume \((x \land y)\) and \((x \land \neg y)\) are mutually exclusive. By (3), \(\Pr(x \land y) + \Pr(x \land \neg y)\) is equivalent to \(\Pr((x \land y) \lor \Pr(x \land \neg y))\). By logic, \((x \land y) \lor (x \land \neg y)\) is equivalent to \((x \lor (y \lor \neg y))\), and, by problem 17, \(\Pr((x \land y) \lor (x \land \neg y)) = \Pr(x \lor (y \lor \neg y))\). \(y \lor \neg y\) is a \(T\), so \(\Pr(x \lor (y \lor \neg y)) = \Pr(x \lor T)\). Since the conjunction of any proposition \(p\) and a tautology is equivalent to the proposition \(p\), this implies that \(\Pr(x \land T) = \Pr(x)\).

Q.E.D.

**Def'n:** \(x\) and \(y\) are probabilistically independent of each other when \(\Pr(x \land y) = \Pr(x) \cdot \Pr(y)\).
20. Prove, axiomatically, that when $x$ and $y$ are independent and $\Pr(x) > 0$ and $\Pr(y) > 0$:
   (a) $\Pr(x \mid y) = \Pr(x)$
   (b) $\Pr(y \mid x) = \Pr(y)$

(a) **Proof**: Assume that $x$ and $y$ are independent, and that $\Pr(x) > 0$ and $\Pr(y) > 0$; $\Pr(x \mid y) = (\Pr(x \& y) / \Pr(y))$. By definition, $\Pr(x \& y) = \Pr(x) \cdot \Pr(y)$ when $x$ and $y$ are independent. By problem 17 we can substitute equivalent propositions in probability statements. So $\Pr(x \mid y) = ([\Pr(x) \cdot \Pr(y)] / \Pr(y)]$. By algebra the $\Pr(y)$'s will cancel out, leaving $\Pr(x \mid y) = \Pr(x)$.

   Q.E.D.

(b) (Proof will be very similar to the proof in a.)

21. Prove that if $x$, $y$, and $z$ are independent, then: $\Pr(x \& (y \lor z)) = \Pr(x) \cdot \Pr(y \lor z)$

A general form of (3) is that $\Pr(p \lor q) = \Pr(p) + \Pr(q) - \Pr(p \& q)$.

Assume that $x$, $y$, and $z$ are mutually independent. Then:

$\Pr(x \& (y \lor z)) = \Pr((x \& y) \lor (x \& z))$. (logic)

$= \Pr(x \& y) + \Pr(x \& z) - \Pr(x \& y \& z)$ (Generalized 3)

$= \Pr(x \& y) + \Pr(x \& z) - \Pr(x \& y \& z)$ (logic)

$= \Pr(x) \cdot \Pr(y) + \Pr(x) \cdot \Pr(z) - \Pr(x) \cdot \Pr(y \& z)$ (independence)

$= \Pr(x) \cdot \Pr(y) + \Pr(z) - \Pr(y \& z)$ (algebra)

$= \Pr(x) \cdot \Pr(y \lor z)$ (Generalized 3)

Q.E.D.
Use Dutch Book proofs to demonstrate that for an agent’s beliefs to be coherent, they must be consistent with the following axioms/theorems/definitions of probability.

In all cases, the following will hold:

Mr. B will select a betting quotient \( q \) on an event \( E \) occurring (sometimes denoted as \( q(E) \)). Ms. A will then choose a stake \( S \). Mr. B must pay Ms. A \( qS \); if \( E \) occurs, then Ms. A will pay Mr. B stake \( S \). \( S \) may be positive (+ve) or negative (-ve). In these cases, \( q \) is taken to be equivalent to the degree of belief that \( E \) will occur.

So, if \( E \) occurs, then Ms. A’s gain \( (G) \) will be \( qS \) minus \( S \), i.e., \( G = qS - S = S(q - 1) \). If \( E \) fails to occur, then Ms. A’s gain \( (G) \) will simply be \( qS \).

In case \( G \) is positive regardless of whether or not \( E \) occurs, then Mr. B is subject to a Dutch Book. Ms. A will always win the bet, and Mr. B will always lose.

(Note: rather than betting on an event occurring, you could bet on the truth of proposition \( p \). This shifts the degrees of belief from being over events to over propositions.)

22. \( 0 \leq \Pr(E) \leq 1 \)

(i) Demonstration that inconsistency with (22) is not coherent, i.e., it would leave an agent subject to a Dutch Book.

There are two ways that Mr. B might select a betting quotient \( q \) that was inconsistent with (22). He might select a betting quotient less than 0, or greater than 1. Let’s look at each in turn, demonstrating how either sets Mr. up for a Dutch Book.

(i) Suppose Mr. B set \( q < 0 \). If Ms. A sets \( S < 0 \), Mr. B will be subject to a Dutch Book.

**Short version:**

If \( E \),

\[ G = S(q - 1); \]
\[ q - 1 < 0; S < 0; \therefore G > 0 \]

If \( \sim E \),

\[ G = qS; \]
\[ q < 0; S < 0; \therefore G > 0 \]

\[ \therefore q < 0 \supset G > 0 \]

(If \( q < 0 \), then \( G > 0 \))

**Long version:**
If $E$ occurs, then $G = S(q - 1)$. $q - 1$ will be negative, since $q$ is negative. Multiplying that by $S$ is to multiply a negative number by a negative number, producing a positive result, and, a positive gain for Ms. A. So if $E$ occurs, Ms. A will always win.

If $E$ fails to occur, then $G = qS$. Both $q$ and $S$ are negative, so the product would be positive, again producing a positive gain for Ms. A. So if $E$ fails to occur, Ms. A will always win.

So if Mr. B sets $q < 0$, then regardless of whether $E$ occurs or not Ms. A will net a positive gain if she sets $S < 0$. Thus, Mr. B would be subject to a Dutch Book.

(ii) Suppose Mr. B sets $q > 1$. If Ms. A sets $S > 0$, Mr. B will be subject to a Dutch Book.

**Short version:**

If $E$,

\[
G = S(q - 1);
q - 1 > 0;
S > 0;
\therefore G > 0
\]

If $\neg E$,

\[
G = qS;
q > 0;
S > 0;
\therefore G > 0
\]

\[\therefore q > 1 \supset G > 0\]

(If $q > 1$, then $G > 0$)

**Long version:**

If $E$ occurs, then $G = S(q - 1)$. $q - 1$ will be positive, since $q > 1$. Multiplying that by $S$ is to multiply a positive number by a positive number, producing a positive result, and, a positive gain for Ms. A. So if $E$ occurs, Ms. A will always win.

If $E$ fails to occur, then $G = qS$. Both $q$ and $S$ are positive, so the product would be positive, again producing a positive gain for Ms. A. So if $E$ fails to occur, Ms. A will always win.

So if Mr. B sets $q > 1$, then regardless of whether $E$ occurs or not Ms. A will net a positive gain if she sets $S > 0$. Thus, Mr. B would be subject to a Dutch Book.

\[\therefore \text{So, by (i) and (ii), if Mr. B fails to set a betting quotient that satisfies (22), then his degrees of belief are not coherent.}\]
(II) Demonstration that satisfying (22) produces coherence; i.e., an agent with coherent beliefs may not have a Dutch Book set against them.

For Mr. B to satisfy (22), he must select a betting quotient $q$ such that $0 \leq q \leq 1$. If Mr. B selects $0 \leq q \leq 1$, then Ms. A cannot select a stake $S$ so that she will always win.

Recall that Ms. A’s gain ($G$) is the following:
- if $E$ occurs, $G = qS - S = S(q - 1)$;
- if $E$ does not occur, $G = qS$.

If Ms. A sets $S > 0$,
- If $E$, $G = S(q - 1)$;
  - $q - 1 < 0$;
  - $S > 0$;
  - $\therefore G < 0$
- If $\neg E$, $G = qS$;
  - $q > 0$;
  - $S > 0$;
  - $\therefore G > 0$
- $\therefore G > 0$ is contingent on $E$.

If Ms. A sets $S < 0$,
- If $E$, $G = S(q - 1)$;
  - $q - 1 < 0$;
  - $S < 0$;
  - $\therefore G > 0$
- If $\neg E$, $G = qS$;
  - $q > 0$;
  - $S < 0$;
  - $\therefore G < 0$
- $\therefore G > 0$ is contingent on $E$.

**Prose:**
If Ms. A sets $S > 0$, $G$ will be positive if $q \neq 0$ and $E$ fails to occur, but will be negative if $E$ occurs.
If Ms. A sets $S < 0$, Ms. A will make money if $q \neq 1$ and $E$ occurs, but will lose money if $E$ does not occur.

$\therefore$ If Mr. B’s betting quotients satisfy (22), then his degrees of belief are coherent.
23. Demonstrate that if an agent fails to satisfy the following theorem that they will not have coherent beliefs:
\[ \Pr(E) = 1 - \Pr(\neg E). \]

(*Note:* an easy way to do this is to note that \( E \lor \neg E \) is a tautology, and by the Additivity Axiom (axiom 3) and Certainty Axiom (axiom 2), \( \Pr(E \lor \neg E) = \Pr(E) + \Pr(\neg E) = 1 = \Pr(T) \). 23 yields ‘\( \Pr(E) + \Pr(\neg E) = 1 \’ from simple algebra, and, using substitution, you may proceed with your demonstration by showing that inconsistency with ‘\( \Pr(T) = 1 \’ is not coherent. Here I will proceed with straight demonstration of 23 as presented, for instructive purposes.)

There are two ways that Mr. B might select betting quotients \( q \) that were inconsistent with (23). He might select betting quotients so that \( q(E) > 1 - q(\neg E) \), or so that \( q(E) < 1 - q(\neg E) \). Let’s look at each in turn, demonstrating how either sets Mr. B up for a Dutch Book.

(i) Suppose Mr. B sets betting quotients \( q(E) \) and \( q(\neg E) \) such that \( q(E) > 1 - q(\neg E) \).
If Ms. A sets \( S(E) = S(\neg E) \), and \( S > 0 \), Mr. B will be subject to a Dutch Book.

**Short version:**

If \( E \),
\[
G = q(E)S - S + q(\neg E)S = S[q(E) + q(\neg E) - 1];
\]
\[
q(E) + q(\neg E) > 1;
\]
\[
\therefore q(E) + q(\neg E) - 1 > 0;
\]
\[
S > 0;
\]
\[
\therefore G > 0
\]

If \( \neg E \),
\[
G = q(E)S + q(\neg E)S - S = S(q(E) + q(\neg E) - 1);\]
\[
q(E) + q(\neg E) > 1;
\]
\[
\therefore q(E) + q(\neg E) - 1 > 0;
\]
\[
S > 0;
\]
\[
\therefore G > 0
\]

\[
\therefore q(E) > 1 - q(\neg E) \supset G > 0
\]
(If \( q(E) > 1 - q(\neg E) \), then \( G > 0 \))

**Long version:**

There are two possible outcomes, either \( E \) occurs, or \( \neg E \) occurs. (Leave aside whether we ought to treat this as betting on a single event occurring. See note above on \( \Pr(T) = 1 \).) Recall that for each bet, Mr. B must pay \( q(X)S \) to play, and receives \( S \) if \( X \) occurs. In this case, Mr. B must pay for two betting quotients. Since either \( E \) or \( \neg E \) must occur, but both may not occur, he will get one of his bets paid off. On either outcome, Ms. A is guaranteed a positive gain \( G \).
A quick bit of algebra will make this easier. Since $q(E) > 1 - q(\neg E)$, we also know that $q(E) + q(\neg E) > 1$. We deduce this by adding $q(\neg E)$ to each side of the first equation.

If $E$ occurs, $G = q(E)S - S + q(\neg E)S$. In order, that is Mr. B’s payment to Ms. A to bet on $E$, Mr. B’s payoff from $E$’s occurrence, and Mr. B’s payment to bet on $\neg E$. We can factor out $S$ using algebra to get $G = S[q(E) - 1 + q(\neg E)]$. A little rearrangement yields: $G = S[q(E) + q(\neg E) - 1]$. Since ‘$q(E) + q(\neg E) > 1$’, ‘$q(E) + q(\neg E) - 1$’ will always be positive. So, if $S > 0$, then $G > 0$, because a positive number multiplied by a positive number yields a positive outcome. So if $E$ occurs, Ms. A will always win.

If $E$ fails to occur (i.e., $\neg E$ occurs), $G = q(E)S + q(\neg E)S - S$. Factoring out $S$ yields $G = S[q(E) + q(\neg E) - 1]$. Since ‘$q(E) + q(\neg E) > 1$’, ‘$q(E) + q(\neg E) - 1$’ will always be positive. So, if $S > 0$, then $G > 0$, because a positive number multiplied by a positive number yields a positive outcome. So if $E$ fails to occur, Ms. A will always win.

So if Mr. B sets betting quotients such that $q(E) > 1 - q(\neg E)$, then regardless of whether $E$ occurs or not Ms. A will net a positive gain if she sets $S(E) = S(\neg E)$, and $S > 0$. Thus, Mr. B would be subject to a Dutch Book.

(ii) Suppose Mr. B sets betting quotients $q(E)$ and $q(\neg E)$ such that $q(E) < 1 - q(\neg E)$. If Ms. A sets $S(E) = S(\neg E)$, and $S < 0$, Mr. B will be subject to a Dutch Book.

**Short version:***

If $E$,

$$G = q(E)S - S + q(\neg E)S = S[q(E) + q(\neg E) - 1];$$

$$q(E) + q(\neg E) < 1;$$

$$\therefore q(E) + q(\neg E) - 1 < 0;$$

$$S < 0;$$

$$\therefore G > 0$$

If $\neg E$,

$$G = q(E)S + q(\neg E)S - S = S[q(E) + q(\neg E) - 1];$$

$$q(E) + q(\neg E) < 1;$$

$$\therefore q(E) + q(\neg E) - 1 < 0;$$

$$S < 0;$$

$$\therefore G > 0$$

$$\therefore q(E) > 1 - q(\neg E) \implies G > 0$$

(If $q(E) > 1 - q(\neg E)$, then $G > 0$)

**Long version:**

There are two possible outcomes, either $E$ occurs, or $\neg E$ occurs. Recall that for each bet, Mr. B must pay $q(X)S$ to play, and receives $S$ if $X$ occurs. In this case, Mr. B must pay for two betting quotients. Since either $E$ or $\neg E$ must occur, but both may not
occur, he will get one of his bets paid off. On either outcome, Ms. A is guaranteed a positive gain \( G \).

A quick bit of algebra will make this easier. Since \( q(E) < 1 - q(\neg E) \), we also know that \( q(E) + q(\neg E) < 1 \). We deduce this by adding \( q(\neg E) \) to each side of the first equation.

If \( E \) occurs, \( G = q(E)S - S + q(\neg E)S \). We can factor out \( S \) using algebra to get \( G = S[q(E) - 1 + q(\neg E)] \). A little rearrangement yields: \( G = S[q(E) + q(\neg E) - 1] \). Since \( q(E) + q(\neg E) < 1 \), \( q(E) + q(\neg E) - 1' \) will always be negative. So, if \( S < 0 \), then \( G > 0 \), because a negative number multiplied by a negative number yields a positive outcome. So if \( E \) occurs, Ms. A will always win.

If \( E \) fails to occur (i.e., \( \neg E \) occurs), \( G = q(E)S + q(\neg E)S - S \). Factoring out \( S \) yields \( G = S[q(E) + q(\neg E) - 1] \). Since \( q(E) + q(\neg E) < 1 \), \( q(E) + q(\neg E) - \neg E \) will always be negative. So, if \( S > 0 \), then \( G > 0 \), because a negative number multiplied by a negative number yields a positive outcome. So if \( E \) fails to occur, Ms. A will always win.

So if Mr. B sets betting quotients such that \( q(E) < 1 - q(\neg E) \), then regardless of whether \( E \) occurs or not Ms. A will net a positive gain if she sets \( S(E) = S(\neg E) \), and \( S < 0 \). Thus, Mr. B would be subject to a Dutch Book.

\[ \therefore \text{So, by (i) and (ii), if Mr. B fails to set a betting quotient that satisfies (23), then his degrees of belief are not coherent.} \]

24. Demonstrate that if an agent fails to satisfy the Additivity Axiom that they will not have coherent beliefs:
\[ \Pr(X \lor Y) = \Pr(X) + \Pr(Y); \] where \( X \) and \( Y \) are mutually exclusive events.

There are two ways that Mr. B might select betting quotients that were inconsistent with the additivity axiom, namely \( q(X \lor Y) > q(X) + q(Y) \); or \( q(X \lor Y) < q(X) + q(Y) \). Let’s look at both in turn.

(i) Suppose Mr. B sets betting quotients such that \( q(X \lor Y) > q(X) + q(Y) \). Let’s call these \( q_1 \), \( q_2 \), and \( q_3 \), respectively, i.e., \( q_1 > q_2 + q_3 \).

If Ms. A sets \( S_1 > 0 \), \( S_2 = S_3 \), and \( S_1 = -S_2 \) (and, by substitution, \( S_1 \) will also be equal to the negation of \( S_2 \)), then Mr. B will be subject to a Dutch Book.

**Proof:**

If \( \neg X \& \neg Y \),
\[ G = q_1S + q_3S + q_3S = S(q_1 - q_2 - q_3); \]
\[ q_1 > q_2 + q_3; \]
\[ \therefore q_1 - q_2 - q_3 > 0; \]
\[ S > 0; \]
\[ \therefore G > 0 \]
If \( X \& \neg Y \),
\[ G = q_3S - S + q_2S + q_2S = S(q_1 - 1 - q_2 + 1 - q_3) = S(q_1 - q_2 - q_3); \]
\[ q_1 > q_2 + q_3; \]
\[ \therefore G > 0 \]
If \( \neg X \& Y \),
\[ G = q_2S + q_3S - S + q_3S = S(q_1 - q_2 - q_3); \]
\[ q_1 > q_2 + q_3; \]
\[ \therefore G > 0 \]
If \( X \& Y \),
\[ G = q_1S + q_3S - S + q_3S = S(q_1 - 1 - q_2 + 1 - q_3) = S(q_1 - q_2 - q_3); \]
\[ q_1 > q_2 + q_3; \]
\[ \therefore G > 0 \]
\[ q_1 > q_2 + q_3; \]
\[ \therefore q_1 - q_2 - q_3 > 0; \]
\[ S > 0; \]
\[ \therefore G > 0 \]

If \( \sim X & Y \),
\[ G = q_1 S + q_2 (-S) + q_3 (-S) = S(q_1 - 1 - q_2 - q_3 + 1) = S(q_1 - q_2 - q_3); \]
\[ q_1 > q_2 + q_3; \]
\[ \therefore q_1 - q_2 - q_3 > 0; \]
\[ S > 0; \]
\[ \therefore G > 0 \]

Because \( X \) and \( Y \) are mutually exclusive, \( X & Y \) is not possible. (Though you could demonstrate that if it were, the proof still holds. Try it!)

(ii) Suppose Mr. B sets betting quotients such that \( q(X v Y) < q(X) + q(Y) \). Let's call these \( q_1, q_2, \) and \( q_3 \), respectively, i.e., \( q_1 < q_2 + q_3 \).

If Ms. A sets \( S_1 < 0, S_2 = S_3, \) and \( S_1 = -S_2 \) (and, by substitution, \( S_1 \) will also be equal to the negation of \( S_3 \)), then Mr. B will be subject to a Dutch Book.

**Proof:**

If \( \sim X & \sim Y \),
\[ G = q_1 S + q_2 (-S) + q_3 (-S) = S(q_1 - q_2 - q_3); \]
\[ q_1 < q_2 + q_3; \]
\[ \therefore q_1 - q_2 - q_3 < 0; \]
\[ S < 0; \]
\[ \therefore G > 0 \]

If \( X & \sim Y \),
\[ G = q_1 S - S + q_2 (-S) - (-S) + q_3 (-S) = S(q_1 - 1 - q_2 + 1 - q_3) = S(q_1 - q_2 - q_3); \]
\[ q_1 < q_2 + q_3; \]
\[ \therefore q_1 - q_2 - q_3 < 0; \]
\[ S < 0; \]
\[ \therefore G > 0 \]

If \( \sim X & Y \),
\[ G = q_1 S - S + q_2 (-S) + q_3 (-S) - (-S) = S(q_1 - 1 - q_2 - q_3 + 1) = S(q_1 - q_2 - q_3); \]
\[ q_1 < q_2 + q_3; \]
\[ \therefore q_1 - q_2 - q_3 < 0; \]
\[ S < 0; \]
\[ \therefore G > 0 \]

\[ \therefore \] So, by (i) and (ii), if Mr. B fails to set a betting quotient that satisfies the Additivity axiom, then his degrees of belief are not coherent.