Supplemental Homework
(Due in class, Thursday Dec. 8)

Section 1 – Probability Space

1. What are all the possible atomic propositions or states of:
   (a) Boolean algebra 1 \((B_1)\) consisting of two atomic sentence-letters \((X, Y)\)?
   (b) \(B_2\) consisting of three atomic sentence-letters \((X, Y, Z)\)?

2. Display the possible logical space of \(B_1 \& B_2\) using truth tables and Venn diagrams.

3. What is a probability model \((M)\)?

4. We learned that Probability Models \((M)\) may be conceptualized as extensions of Boolean Algebras \((B)\). Consider a \(B\) with two atomic sentence letters \((X, Y)\), and four states:
   \((s_1): (X \& Y)\)
   \((s_2): (X \& \sim Y)\)
   \((s_3): (\sim X \& Y)\)
   \((s_4): (\sim X \& \sim Y)\)
   (a) Display the probability space of \(M\) of \(B\) using truth tables and Venn diagrams. For the rest of the homework assignment, this will be \(M_1\).
   (b) Display the probability space of \(M\) of \(B\) using truth tables and Venn diagrams when the values of \(\Pr_M(s_i)\) are:
      \(\Pr_M(s_1) = 1/6\)
      \(\Pr_M(s_2) = 1/4\)
      \(\Pr_M(s_3) = 1/8\)
      \(\Pr_M(s_4) = 11/24\)
      For the rest of the homework assignment, this will be \(M_3\).
   (c) Display the probability space of \(M\) of \(B\) using truth tables and Venn diagrams when the values of \(\Pr_M(s_i)\) are:
      \(\Pr_M(s_1) = 0\)
      \(\Pr_M(s_2) = 1/3\)
      \(\Pr_M(s_3) = 1/6\)
      \(\Pr_M(s_4) = 1/2\)
      For the rest of the homework assignment, this will be \(M_4\).
   (d) Display the probability space of \(M\) of \(B_2\) from problem 1b above using truth tables and Venn diagrams. For the rest of the homework assignment, this will be \(M_2\).

5. (a) What is the relation between \(X\) and \(Y\) in \(M_4\)?
   (b) Assign probabilities to the states of \(M_3\) such that \(X\) and \(Y\) will be exclusive and exhaustive. For the rest of the homework assignment this will be \(M_5\).
   (c) Draw a truth table and Venn diagram displaying the probability space of \(M_5\) in 5b.
Section 2 – Algebraic Proofs

6. (a) What is the probability of a proposition \( p \) in a probability model \( M \)?
(b) What is the probability of \( (X \lor Y) \) in \( M \)?
(c) What is the probability of \( (X \land Y) \) in \( M \)?
(d) What is the probability of \( (X \lor Y) \) in \( M \)?
(e) What is the probability of \( (X \lor Y) \) in \( M \)?

7. Determine the probability of the following propositions \( (p) \) in \( M_1, M_3, \) and \( M_4 \): 
   (a) \( (X \land Y) \)
   (b) \( (\neg X \lor Y) \)
   (c) \( (X \rightarrow Y) \land Y \)

8. Determine the probability of the following propositions \( (p) \) in \( M_2 \): 
   (a) \( (X \lor Y) \)
   (b) \( (X \land Y \land Z) \)
   (c) \( (X \lor Y) \rightarrow Z \)
   (d) \( (X \lor Y) \land (Y \lor Z) \)

9. Prove, algebraically, that: \( \Pr(X \lor Y) = \Pr(X) + \Pr(Y) - \Pr(X \land Y) \) in (a) \( M_1 \) and (b) \( M_4 \).

10. Prove, algebraically, that: \( \Pr(X) = \Pr(X \land Y) + \Pr(X \land \neg Y) \) in \( M_1 \).

11. Define \( \Pr(X \mid Y) \) in a non-conditional form.

12. Prove, algebraically, that: \( \Pr(X \mid Y) = \frac{\Pr(Y \mid X) \cdot \Pr(X)}{\Pr(Y)} \) in \( M_1 \).

13. Prove, algebraically, that: \( \Pr(X \lor Y \mid Z) = \Pr(X \mid Z) + \Pr(Y \mid Z) - \Pr(X \land Y \mid Z) \) in \( M_2 \).

14. What is Bayes’ Rule?

15. Prove, algebraically, that \( \Pr(x \land y) = \Pr(x) \cdot \Pr(y) \equiv \Pr(x \mid y) = \Pr(x \mid \neg y) \).
   (hint: Do this in \( M_1 \); start by determining what each sentence is algebraically, then what each side of the biconditional would look like.)
A probability model \((M)\) is a Boolean algebra\(^1\) of propositions \(B\) together with a probability function \(\Pr(\cdot)\) satisfying the following three axioms:

1. For any proposition, the probability of that proposition is non-negative;
   \[ \forall p, \Pr(p) \geq 0 \]

2. All tautologies \((T)\) have a probability of 1;
   \[ \Pr(T) = 1; \]
   where \(\Omega\) is a logical truth, \(\Pr(\Omega) = 1\)

3. The probability of a disjunction of propositions is the sum of the probability of its disjuncts minus the probability of the conjunction of its disjuncts:
   \[ \Pr(p \lor q) = \Pr(p) + \Pr(q) - \Pr(p \land q) \]

A special case of this axiom concerns the disjunction of mutually exclusive (m.e.) propositions, in which case the probability is simply the sum of the probability of its disjuncts (why?).

\[ \Pr(p \lor q) = \Pr(p) + \Pr(q) \]

**Def'n:**

\[ \Pr(X \mid Y) = \frac{\Pr(X \land Y)}{\Pr(Y)} = \Pr\left( \frac{X \land Y}{Y} \right) \]

16. Prove, axiomatically, that \(\Pr(\sim p) = 1 - \Pr(p)\). This will be theorem 4.

17. Prove, axiomatically, that the probability of a logical falsity is 0, i.e., \(\Pr(F) = 0\).

18. Prove, axiomatically, that if \(x\) and \(y\) mutually entail each other, then \(\Pr(x) = \Pr(y)\).

19. Prove, axiomatically, that \(\Pr(x) = \Pr(x \land y) + \Pr(x \land \sim y)\)
   
   (hint: \(x \land T\) is logically equivalent to \(x\). Why?)

**Def'n:** \(x\) and \(y\) are probabilistically independent of each other when \(\Pr(x \land y) = \Pr(x) \cdot \Pr(y)\).

20. Prove, axiomatically, that when \(x\) and \(y\) are independent and \(\Pr(x) > 0\) and \(\Pr(y) > 0\):
   
   (a) \(\Pr(x \mid y) = \Pr(x)\)
   
   (b) \(\Pr(y \mid x) = \Pr(y)\)

21. Prove that if \(x, y,\) and \(z\) are independent, then: \(\Pr(x \land (y \lor z)) = \Pr(x) \cdot \Pr(y \lor z)\)

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\(^1\) This means we get to import the rules of logic and algebra.
Section 4 – Dutch Book Proofs

Use Dutch Book proofs to demonstrate that for an agent’s beliefs to be coherent, they must be consistent with the following axioms/theorems/definitions of probability.

22. $0 \leq \Pr(E) \leq 1$.

(I) Demonstration that inconsistency with (22) is not coherent, i.e., it would leave an agent subject to a Dutch Book.

(II) Demonstration that satisfying (22) produces coherence; i.e., an agent with coherent beliefs may not have a Dutch Book set against them.

23. Demonstrate that if an agent fails to satisfy the following theorem that they will not have coherent beliefs:

$$\Pr(E) = 1 - \Pr(\neg E).$$

24. Demonstrate that if an agent fails to satisfy the following theorem that they will not have coherent beliefs:

$$\Pr(X \lor Y) = \Pr(X) + \Pr(Y);$$

where $X$ and $Y$ are mutually exclusive events.